

Lecture

Discrete-Time Signals and Systems

1. Discrete-Time Signals and Systems

- signal classification -> signals to be applied in digital filter theory within our course,
- some elementary discrete-time signals,
- discrete-time systems: definition, basic properties review, discrete-time system classification, input-output model of discrete-time systems -> system to be applied in digital filter theory within our course,
- Linear discrete-time time-invariant system description in time-, frequency- and transform-domain.

1.1. Basic Definitions

1.1.1. Discrete and Digital Signals

1.1.1.1. Basic Definitions

Signals may be classified into four categories depending on the characteristics of **the time-variable** and **values** they can take:

- continuous-time signals (analogue signals),
- discrete-time signals,
- continuous-valued signals,
- discrete-valued signals.

Continuous-time (analogue) signals:

Time: defined for every value of time $t \in R$,

Descriptions: functions of a continuous variable $t: f(t)$,

Notes: they take on values in the continuous interval $f(t) \in (-a, b)$ for $a, b \rightarrow \infty$.

Note: $f(t) \in C$

$$f(t) = \sigma + j\omega$$

$$\sigma \in (-a, b) \text{ and } \omega \in (-a, b)$$

$$a, b \rightarrow \infty$$

Discrete-time signals:

Time: defined only at discrete values of time: $t = nT$,

Descriptions: sequences of real or complex numbers $f(nT) = f(n)$,

Note A.: they take on values in the continuous interval $f(n) \in (-a, b)$ for $a, b \rightarrow \infty$,

Note B.: sampling of analogue signals:

- sampling interval, period: T ,
- sampling rate: *number of samples per second*,
- sampling frequency (Hz): $f_s = 1/T$.

Continuous-valued signals:

Time: they are defined for every value of time or only at discrete values of time,

Value: they can take on all possible values on finite or infinite range,

Descriptions: functions of a continuous variable or sequences of numbers.

Discrete-valued signals:

Time: they are defined for every value of time or only at discrete values of time,

Value: they can take on values from a finite set of possible values,

Descriptions: functions of a continuous variable or sequences of numbers.

Digital filter theory:

Discrete-time signals:

Definition and descriptions: defined only at discrete values of time and they can take all possible values on finite or infinite range (sequences of real or complex numbers: $f(n)$),

Note: sampling process, constant sampling period.

Digital signals:

Definition and descriptions: discrete-time and discrete-valued signals (i.e. discrete -time signals taking on values from a finite set of possible values),

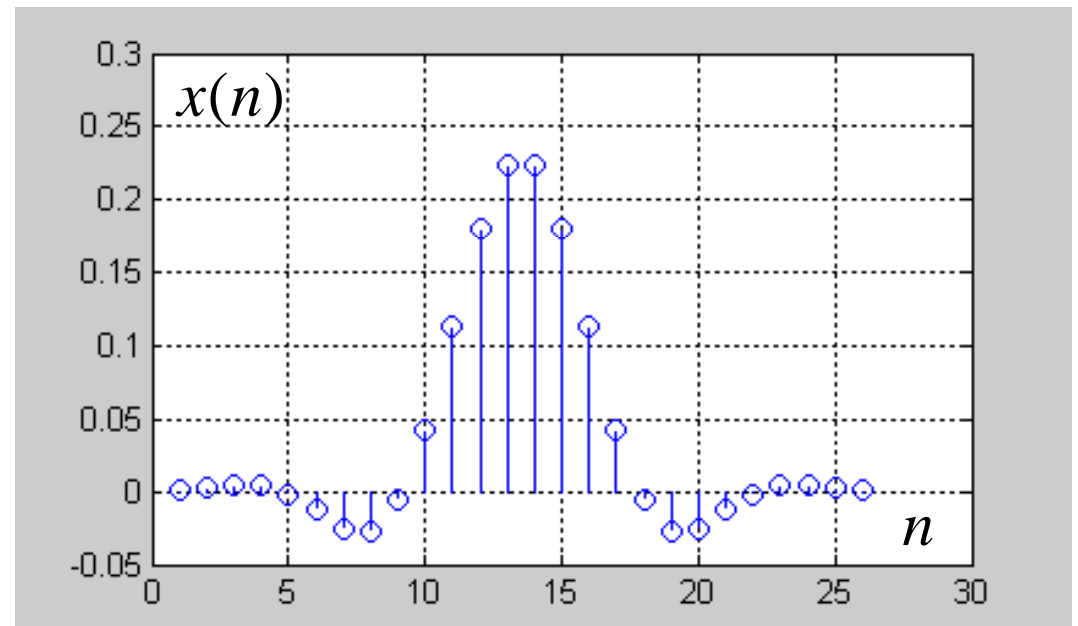
Note: sampling, quantizing and coding process i.e. process of analogue-to-digital conversion. ⁹

1.1.1.2. Discrete-Time Signal Representations

A. Functional representation:

$$x(n) = \begin{cases} 1 & \text{for } n = 1, 3 \\ 6 & \text{for } n = 0, 7 \\ 0 & \text{elsewhere} \end{cases} \quad y(n) = \begin{cases} 0 & \text{for } n < 0 \\ 0,6^n & \text{for } n = 0, 1, \dots, 102 \\ 1 & \text{for } n > 102 \end{cases}$$

B. Graphical representation



C. Tabular representation:

n	...	-2	-1	0	1	2
$x(n)$...	0.12	2.01	1.78	5.23	0.12

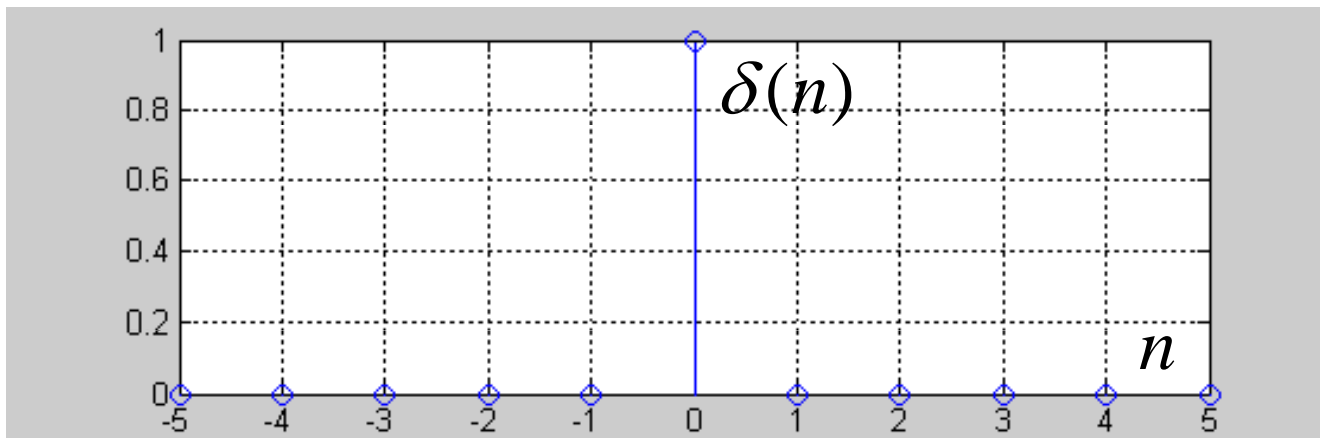
D. Sequence representation:

$$x(n) = \{\dots \quad 0.12 \quad 2.01 \quad 1.78 \quad 5.23 \quad 0.12 \quad \dots\}$$

1.1.1.3. Elementary Discrete-Time Signals

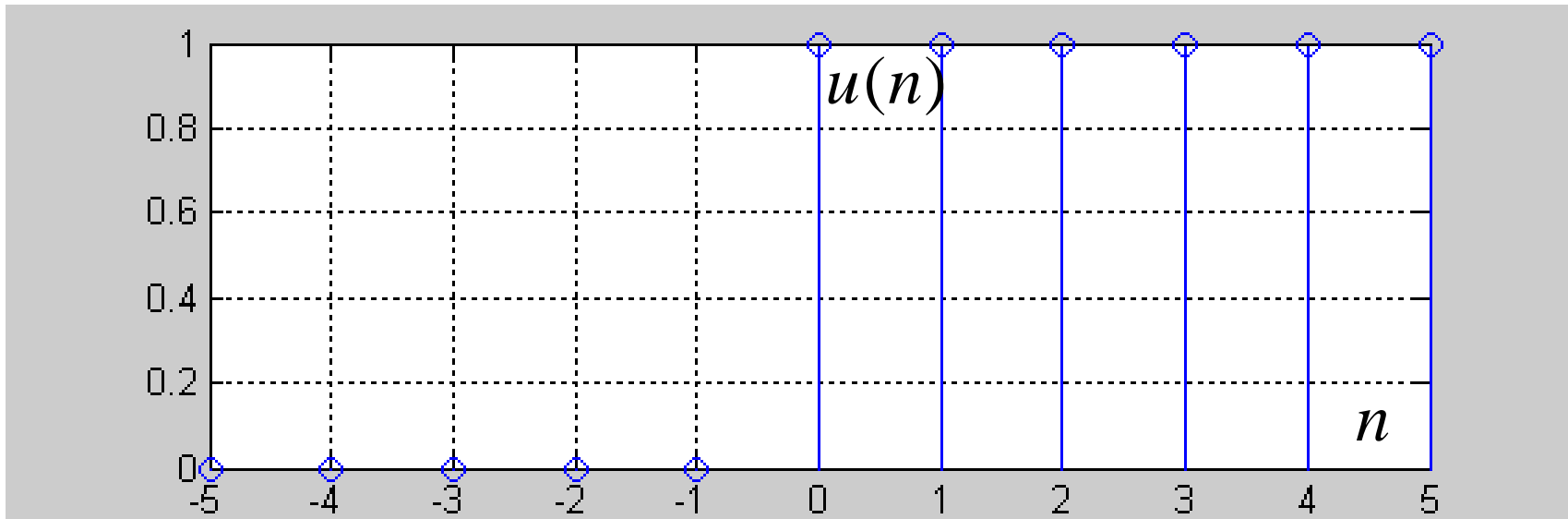
A. Unit sample sequence (unit sample, unit impulse, unit impulse signal)

$$\delta(n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n \neq 0 \end{cases}$$



B. Unit step signal (unit step, Heaviside step sequence)

$$u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$$



C. Complex-valued exponential signal

(complex sinusoidal sequence, complex phasor)

$$x(n) = e^{j\omega nT}, \quad |x(n)| = 1, \quad \arg[x(n)] = \omega nT = 2\pi f \cdot nT = \frac{2\pi f \cdot n}{f_s}$$

where

$\omega \in R$, $n \in N$, $j = \sqrt{-1}$ is imaginary unit

and

T is sampling period and f_s is sampling frequency.

1.1.2. Discrete-Time Systems. Definition

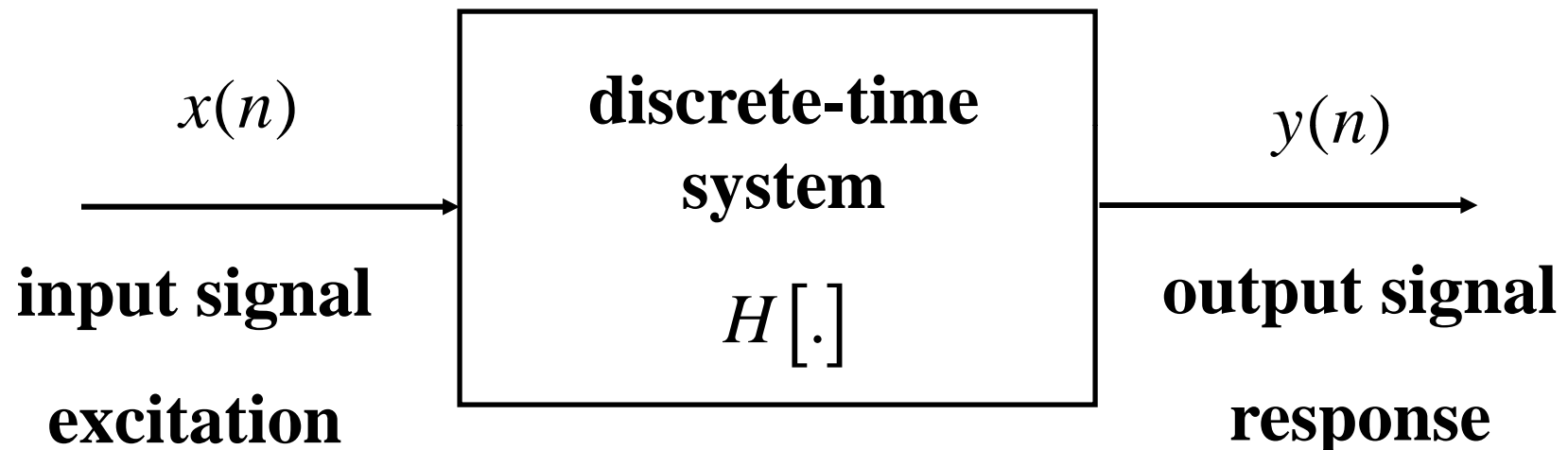
A discrete-time system is a device or algorithm that operates on a discrete-time signal called *the input* or *excitation* (e.g. $x(n)$), according *to some rule* (e.g. $H[.]$) to produce another discrete-time signal called *the output* or *response* (e.g. $y(n)$).

$$y(n) \equiv H[x(n)]$$

This expression denotes also **the transformation $H[.]$ (also called operator or mapping) or processing performed by the system on $x(n)$ to produce $y(n)$.**

Input-Output Model of Discrete-Time System

(input-output relationship description)



$$y(n) \equiv H[x(n)]$$

$$x(n) \xrightarrow{H} y(n)$$

1.1.3. Classification of Discrete-Time Systems

1.1.3.1. Static vs. Dynamic Systems. Definition

A discrete-time system is called *static* or *memoryless* if its output at any time instant n depends on the input sample at the same time, but not on the past or future samples of the input. In the other case, the system is said to be *dynamic* or to have *memory*.

If the output of a system at time n is completely determined by the input samples in the interval from $n-N$ to n ($N \geq 0$), the system is said to have memory of *duration N* .

If $N = 0$, the system is *static* or *memoryless*.

If $0 < N < \infty$, the system is said to have *finite memory*.

If $N \rightarrow \infty$, the system is said to have *infinite memory*.

Examples:

The static (memoryless) systems:

$$y(n) = nx(n) + bx^3(n)$$

The dynamic systems with finite memory:

$$y(n) = \sum_{k=0}^N h(k)x(n-k)$$

The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

1.1.3.2. Time-Invariant vs. Time-Variable Systems.

Definition

A discrete-time system is called *time-invariant* if its input-output characteristics do not change with time. In the other case, the system is called *time-variable*.

Definition. A relaxed system $H[.]$ is *time-* or *shift-invariant* if only if

$$y(n) \equiv H[x(n)] \qquad x(n) \xrightarrow{H} y(n)$$

implies that

$$y(n - k) \equiv H[x(n - k)] \qquad x(n - k) \xrightarrow{H} y(n - k)$$

for *every input signal* $x(n)$ and *every time shift* k .

Examples:

The time-invariant systems:

$$y(n) = x(n) + bx^3(n)$$

$$y(n) = \sum_{k=0}^N h(k)x(n-k)$$

The time-variable systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^N h^{N-n}(k)x(n-k)$$

1.1.3.3. Linear vs. Non-linear Systems. Definition

A discrete-time system is called *linear* if only if it satisfies the *linear superposition principle*. In the other case, the system is called *non-linear*.

Definition. A relaxed system $H[.]$ is *linear* if only if

$$H[a_1x_1(n) + a_2x_2(n)] = a_1H[x_1(n)] + a_2H[x_2(n)]$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2 .

Examples:

The linear systems:

$$y(n) = \sum_{k=0}^N h(k)x(n-k) \quad y(n) = x(n^2) + bx(n-k)$$

The non-linear systems:

$$y(n) = nx(n) + bx^3(n-1) \quad y(n) = \sum_{k=0}^N h(k)x(n-k)x(n-k+1)$$

1.1.3.4. Causal vs. Non-causal Systems. Definition

Definition. A system is said to be *causal* if the output of the system at any time n (i.e., $y(n)$) depends only on present and past inputs (i.e., $x(n)$, $x(n-1)$, $x(n-2)$, ...). In mathematical terms, the output of a *causal* system satisfies an equation of the form

$$y(n) = F [x(n), x(n-1), x(n-2), \dots]$$

where $F[.]$ is some arbitrary function. If a system does not satisfy this definition, it is called *non-causal*.

Examples:

The causal system:

$$y(n) = \sum_{k=0}^N h(k)x(n-k) \quad y(n) = x^2(n) + bx(n-k)$$

The non-causal system:

$$y(n) = nx(n+1) + bx^3(n-1) \quad y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

1.1.3.5. Stable vs. Unstable of Systems. Definitions

An arbitrary relaxed system is said to be ***bounded input - bounded output (BIBO) stable*** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y , such that

$$|x(n)| \leq M_x < \infty \implies |y(n)| \leq M_y < \infty$$

for all n . If for some bounded input sequence $x(n)$, the output $y(n)$ is unbounded (infinite), the system is classified as ***unstable***.

Examples:

The stable systems:

$$y(n) = \sum_{k=0}^N h(k)x(n-k) \quad y(n) = x(n^2) + 3x(n-k)$$

The unstable system:

$$y(n) = 3^n x^3(n-1)$$

1.1.3.6. Recursive vs. Non-recursive Systems.

Definitions

A system whose output $y(n)$ at time n depends on any number of the past outputs values (e.g. $y(n-1)$, $y(n-2)$, ...), is called a ***recursive system***. Then, the output of a causal recursive system can be expressed in general as

$$y(n) = F[y(n-1), y(n-2), \dots, y(n-N), x(n), x(n-1), \dots, x(n-M)]$$

where $F[.]$ is some arbitrary function. In contrast, if $y(n)$ at time n depends only on the present and past inputs

$$y(n) = F[x(n), x(n-1), \dots, x(n-M)]$$

then such a system is called ***nonrecursive***.

Examples:

The nonrecursive system:

$$y(n) = \sum_{k=0}^N h(k)x(n-k)$$

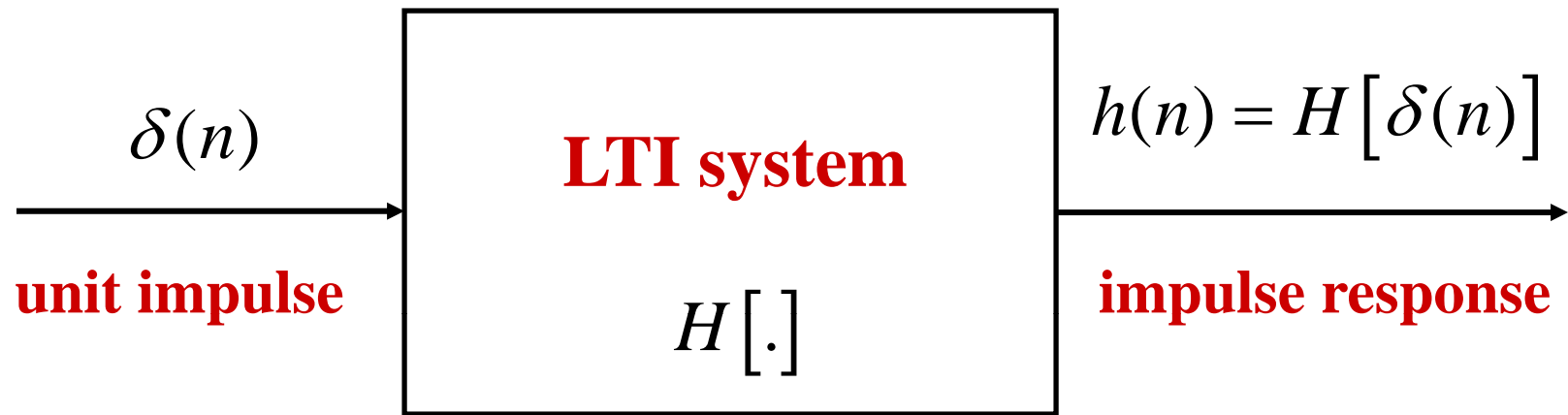
The recursive system:

$$y(n) = \sum_{k=0}^N b(k)x(n-k) - \sum_{k=1}^N a(k)y(n-k)$$

1.2. Linear-Discrete Time Time-Invariant Systems (LTI Systems)

1.2.1. Time-Domain Representation

1.2.1.1 Impulse Response and Convolution



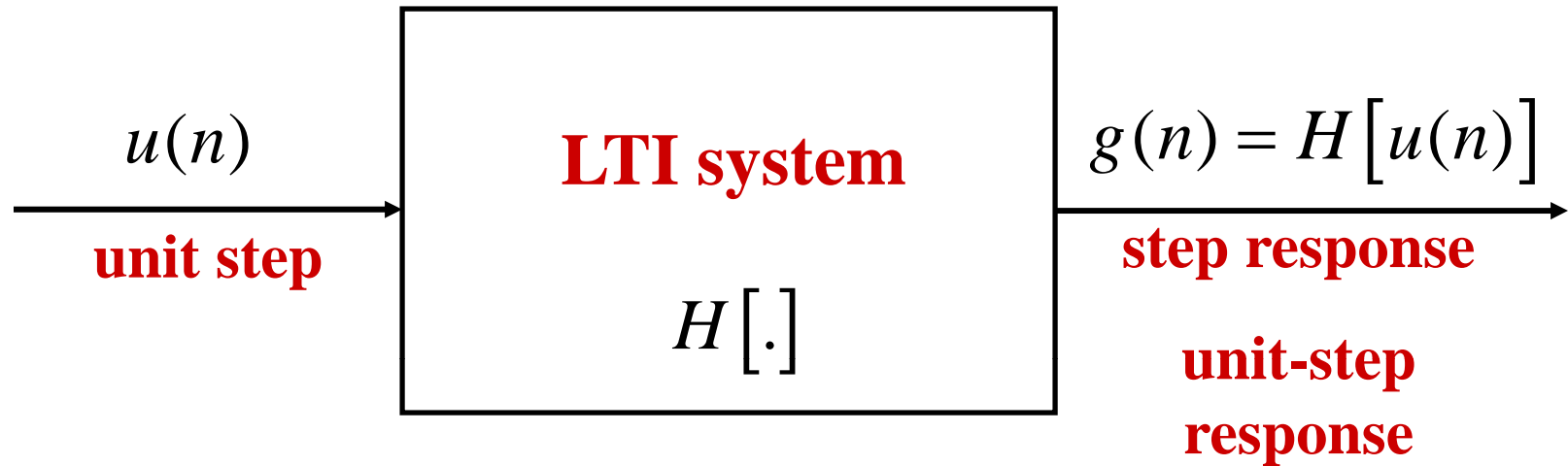
LTI system description by **convolution** (convolution sum):

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = h(n) * x(n) = x(n) * h(n)$$

Four red arrows originate from the word "convolution" in the text above and point to the four convolution sum expressions in the equation above.

Viewed mathematically, the convolution operation satisfies the commutative law.

1.2.1.2. Step Response



$$g(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k) = \sum_{k=-\infty}^n h(k)$$

These expressions relate the impulse response to the step response of the system.

1.2.2. Impulse Response Property and Classification of LTI Systems

1.2.2.1. Causal LTI Systems

A relaxed LTI system is *causal* if and only if its impulse response is zero for negative values of n , i.e.

$$h(n) = 0 \text{ for } n < 0$$

Then, the two equivalent forms of the convolution formula can be obtained for the causal LTI system:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^n x(k)h(n-k)$$

1.2.2.2. Stable LTI Systems

A LTI system is *stable* if its impulse response is absolutely summable, i.e.

$$\sum_{k=-\infty}^{\infty} |h(k)|^2 < \infty$$

1.2.2.3. Finite Impulse Response (FIR) LTI Systems and Infinite Impulse Response (IIR) LTI Systems

Causal **FIR** LTI systems:
$$y(n) = \sum_{k=0}^N h(k)x(n-k)$$

IIR LTI systems:
$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

1.2.2.4. Recursive and Nonrecursive LTI Systems

Causal *nonrecursive* LTI:
$$y(n) = \sum_{k=0}^N h(k)x(n-k)$$

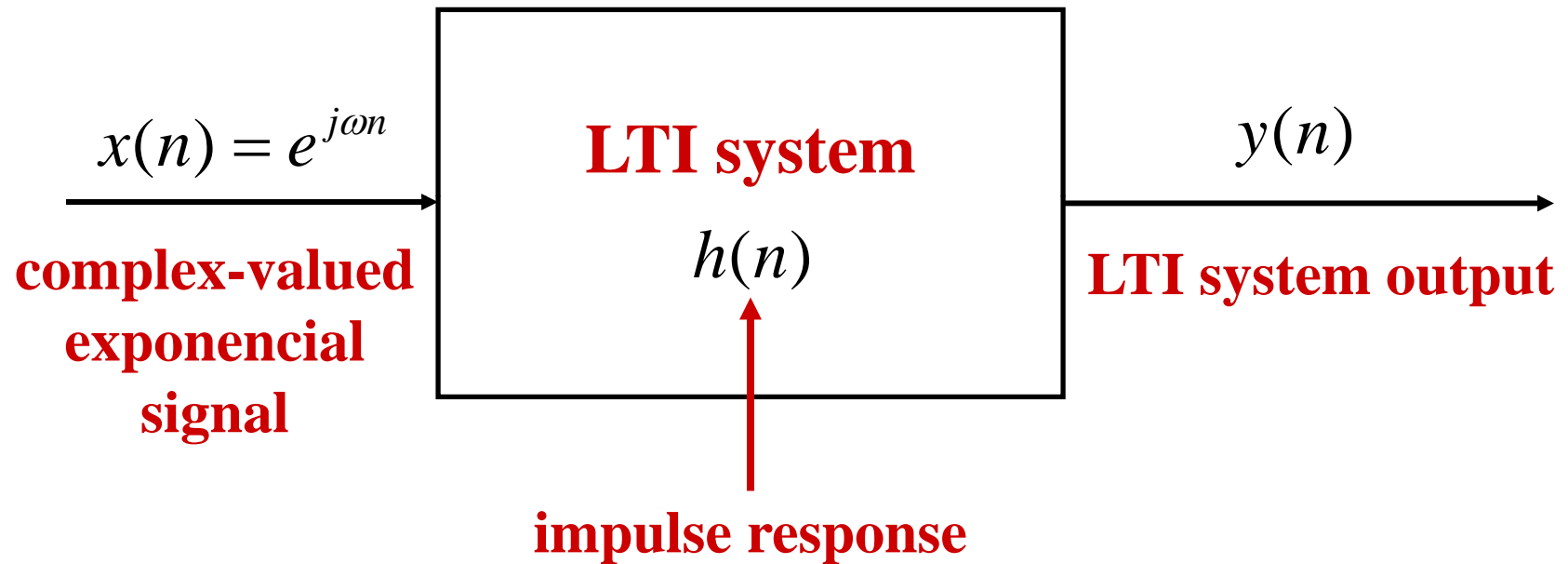
Causal *recursive* LTI:

$$y(n) = \sum_{k=0}^N b(k)x(n-k) - \sum_{k=1}^M a(k)y(n-k)$$

LTI systems:

characterized by *constant-coefficient difference equations*

1.3. Frequency-Domain Representation of Discrete Signals and LTI Systems



$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

LTI system output:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} =$$
$$= \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} e^{j\omega n} = e^{j\omega n} \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = e^{j\omega n} H(e^{j\omega})$$

Frequency response: $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = \operatorname{Re}[H(e^{j\omega})] + j \operatorname{Im}[H(e^{j\omega})]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) \cos \omega k + j \left[- \sum_{k=-\infty}^{\infty} h(k) \sin \omega k \right]$$

$$\operatorname{Re}[H(e^{j\omega})] = \sum_{k=-\infty}^{\infty} h(k) \cos \omega k$$

$$\operatorname{Im}[H(e^{j\omega})] = - \sum_{k=-\infty}^{\infty} h(k) \sin \omega k$$

Magnitude response:

$$|H(e^{j\omega})| = \sqrt{\operatorname{Re}[H(e^{j\omega})]^2 + \operatorname{Im}[H(e^{j\omega})]^2}$$

Phase response:

$$\phi(\omega) = \arg[H(e^{j\omega})] = \operatorname{arctg} \frac{\operatorname{Im}[H(e^{j\omega})]}{\operatorname{Re}[H(e^{j\omega})]}$$

Group delay function:

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$$

1.3.1. Comments on relationship between the impulse response and frequency response

The important property of *the frequency response*

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} = \sum_{k=-\infty}^{\infty} h(k)e^{-j[\omega+2l\pi]} = H(e^{j[\omega+2l\pi]})$$

is fact that this function *is periodic with period 2π* .

In fact, we may view the previous expression as the exponential Fourier series expansion for $H(e^{j\omega})$, with $h(k)$ as the Fourier series coefficients. Consequently, the unit impulse response $h(k)$ is related to $H(e^{j\omega})$ through the integral expression

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

1.3.2. Comments on symmetry properties

For LTI systems with real-valued impulse response, the magnitude response, phase responses, the real component of and the imaginary component of $H(e^{j\omega})$ possess these symmetry properties:

The real component: *even function* of ω periodic with period 2π

$$\operatorname{Re}\left[H(e^{-j\omega})\right] = \operatorname{Re}\left[H(e^{j\omega})\right]$$

The imaginary component: *odd function* of ω periodic with period 2π

$$\operatorname{Im}\left[H(e^{-j\omega})\right] = -\operatorname{Im}\left[H(e^{j\omega})\right]$$

The magnitude response: even function of ω periodic with period 2π

$$\left| H(e^{j\omega}) \right| = \left| H(e^{-j\omega}) \right|$$

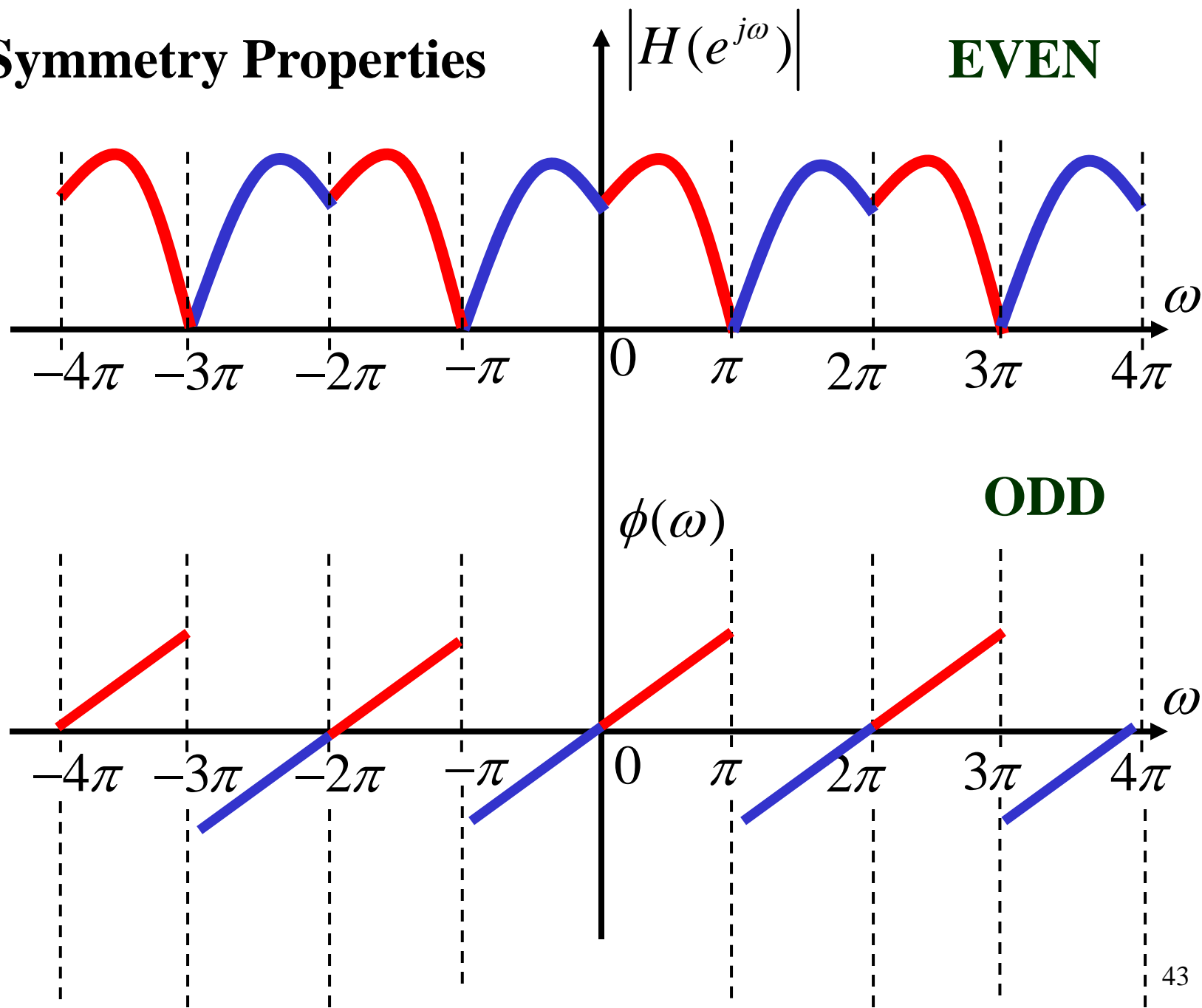
The phase response: odd function of ω periodic with period 2π

$$\arg \left[H(e^{-j\omega}) \right] = -\arg \left[H(e^{j\omega}) \right]$$

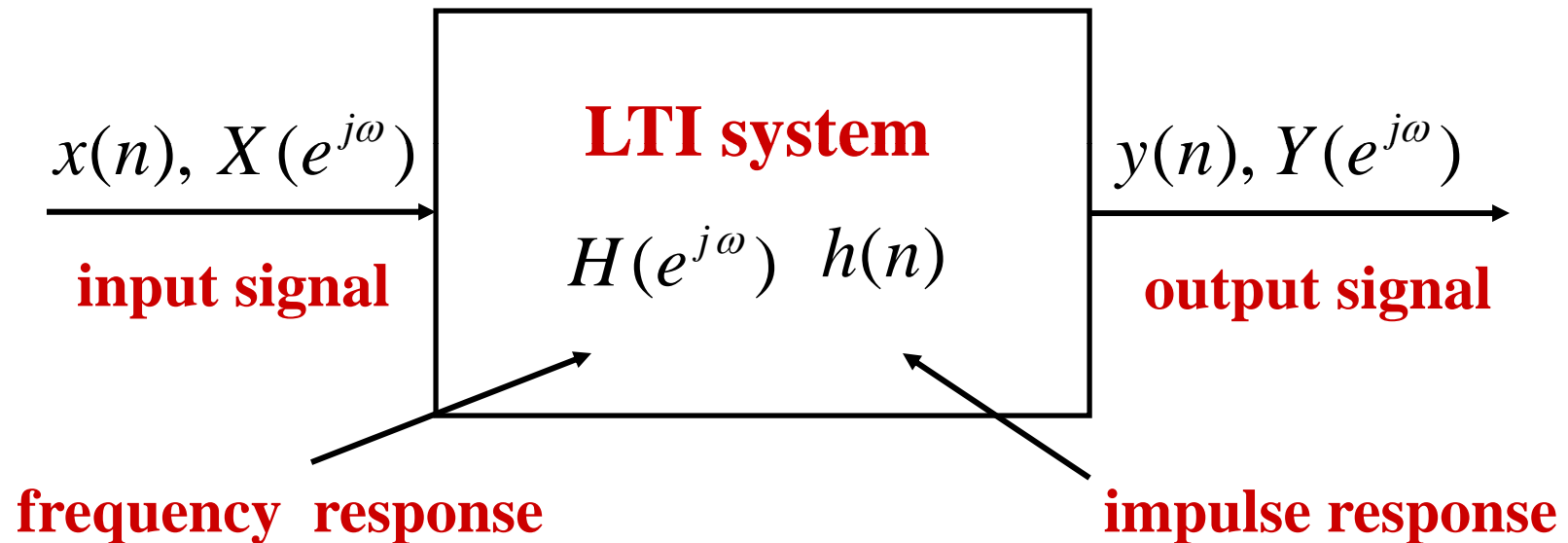
Consequence:

If we know $\left| H(e^{j\omega}) \right|$ and $\phi(\omega)$ for $0 \leq \omega \leq \pi$, we can describe these functions (i.e. also $H(e^{j\omega})$) for all values of ω .

Symmetry Properties



1.3.3. Comments on Fourier Transform of Discrete Signals and Frequency-Domain Description of LTI Systems



The input signal $x(n)$ and the spectrum of $x(n)$:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k} \quad x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega$$

The output signal $y(n)$ and the spectrum of $y(n)$:

$$Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k} \quad y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n} d\omega$$

The impulse response $h(n)$ and the spectrum of $h(n)$:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \quad h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega$$

Frequency-domain description of LTI system:

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

1.3.4. Comments on Normalized Frequency

It is often desirable to express the frequency response of an LTI system in terms of units of frequency that involve sampling interval T . In this case, the expressions:

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \quad h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

are modified to the form:

$$H(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h(kT) e^{-j\omega kT}$$

$$h(nT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} H(e^{j\omega T}) e^{j\omega nT} d\omega$$

$H(e^{j\omega T})$ is periodic with period $2\pi/T = 2\pi F$, where F is sampling frequency.

Solution: **normalized frequency approach**: $F/2 \rightarrow \pi$

Example:

$$F = 100\text{kHz} \quad F/2 = 50\text{kHz} \quad 50\text{kHz} \rightarrow \pi$$

$$f_1 = 20\text{kHz} \quad \omega_1 = \frac{20 \times 10^3}{50 \times 10^3} \pi = \frac{2\pi}{5} = 0.4\pi$$

$$f_2 = 25\text{kHz} \quad \omega_2 = \frac{25 \times 10^3}{50 \times 10^3} \pi = \frac{\pi}{2} = 0.5\pi$$